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A lifting theorem for compact symplectic manifolds

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Abstract. It is shown that the Lie algebra of globally Hamiltonian vector fields on a compact symplectic manifold can be lifted to a Lie algebra of smooth functions on the manifold under Poisson bracket. This implies that any algebra of symmetries of a classical mechanical system described by such a manifold may be realised as an algebra of observables (smooth functions). Parallels between lifting problems in classical and quantum mechanics are explored.

In Hamiltonian mechanics, the relationship between canonical symmetries and constants of the motion is in most respects a very straightforward one. Using the representation of a mechanical system by means of a symplectic structure, that is, an even-dimensional manifold M with symplectic two-form Ω defined on it, one may set up a correspondence between observables (smooth functions on M) and certain vector fields on M. A vector field X on M is said to be globally Hamiltonian if $X \lrcorner \Omega$ is an exact one-form. The particular globally Hamiltonian vector field Γ corresponding to the Hamiltonian function h of the system, which is defined by

$$\Gamma \lrcorner \Omega = -\mathrm{d}h,$$

determines the dynamics of the system. We denote by π the map which associates with each observable f the globally Hamiltonian vector field defined by

$$\pi(f) \lrcorner \Omega = -\mathrm{d}f.$$

Then f is a constant of the motion ($\Gamma(f) = 0$) if and only if $\pi(f)(h) = 0$, and under these circumstances $\pi(f)$ is a canonical symmetry:

$$L_{\pi(f)}\Omega = 0 \qquad \text{and} \qquad [\pi(f), \Gamma] = 0;$$

thus $\pi(f)$ generates transformations which permute the integral curves of Γ.

The Poisson bracket of observables corresponds under π to the Lie bracket of vector fields:

$$\pi(\{f, g\}) = [\pi(f), \pi(g)].$$

If f, g are constants of the motion then so is $\{f, g\}$, and the corresponding symmetry is $[\pi(f), \pi(g)]$. Thus given a Lie algebra of constants of the motion of a classical mechanical system, under Poisson bracket (and one might as well include the Hamiltonian function which will then be an element of the centre of the algebra), there is a corresponding Lie algebra of symmetries, under Lie bracket.

There is one complication in this scheme, and it is to the discussion of it that our paper is devoted. This complication is caused by the fact that to any two observables which differ by a constant function there corresponds one and the same globally Hamiltonian vector field. In other words, π is not an injective map, its kernel being the constant functions. Now one is usually faced with the situation that one knows a Lie algebra of symmetries of a system and wishes to infer something about the corresponding constants of the motion, rather than the other way about. Because of this uncertainty in the choice of an observable to correspond to a globally Hamiltonian vector field, it may not in fact be possible to choose the observables so that they form a Lie algebra even though they correspond to symmetries which do. The problem of finding under what conditions one may make a consistent choice of observables corresponding to elements of a Lie algebra of globally Hamiltonian vector fields so that the observables also form a Lie algebra is an example of a lifting problem, and one speaks of lifting the algebra of vector fields to an algebra of observables.

Lifting problems occur elsewhere in physical contexts, the most familiar example being quantum mechanics. Here the problem arises because the phase space of a quantum mechanical system is, strictly speaking, the space of rays (one-dimensional subspaces) of a complex Hilbert space. A symmetry group will then act as a group of automorphisms of the space of rays, in other words, considered as an abstract group, it will have a 'projective' representation. The lifting problem is to find, if possible, a representation of the group by linear operators on the underlying Hilbert space which covers the projective representation, by making a consistent choice of linear operator for each projective one.

It is perhaps worth reminding the reader of the difference between the Poincaré and the Galilean group in this respect. Every projective representation of the Poincaré group lifts; but this is not true for the Galilean group, and the obstruction to lifting may be identified as the mass of the system. Thus the lifting problem has some considerable physical significance.

The purpose of this paper is to draw out the parallels between the lifting problems in the classical and quantum cases, and to prove that in the classical case of a compact symplectic space every Lie algebra of globally Hamiltonian vector fields lifts to a Lie algebra of observables. This result is of technical interest because its proof makes no use of any special properties of the algebra (it does not depend on analysing its central extensions, for example). Of course, the assumption of compactness usually enables one to draw some useful conclusions, and we shall discuss some quantum mechanical examples; but in many cases of interest in Hamiltonian mechanics the energy surfaces are compact, so there should be opportunities for the application of the result.

We begin by describing the lifting problem in more detail, looking first at the quantum mechanical case since it is probably the more familiar. As we mentioned earlier, the phase space of a quantum mechanical system is usually taken to be the space $\hat{H}$ of rays of a complex Hilbert space H; $\hat{H}$ is given a metric (one minus the probability function), and the automorphism group of $\hat{H}$ is, by Wigner's theorem, isomorphic to the group $\hat{U}(H)$ of unitary operator rays of H. Thus, if $U(H)$ denotes the group of unitary operators in H, $\hat{U}(H)$ is the space of orbits of $U(1)$ (the group of complex numbers of unit modulus) in $U(H)$ under the action $U(1)\times U(H)\to U(H)$ given by

$$(\zeta, u)\to \zeta u \qquad (|\zeta|=1, u\in U(H)).$$

This gives the exact sequence of homomorphisms (π is the canonical projection)

$$1 \to U(1) \to U(H) \xrightarrow{\pi} \hat{U}(H) \to 1.$$

(Strictly speaking, $U(H)$ should also include the anti-unitary operators, but in the present context it is safe to ignore them.)

A topological group G is a symmetry group of $\hat{H}$ if it acts as automorphisms on $\hat{H}$; that is, if $\hat{H}$ carries a (without loss of generality, faithful) projective representation $\hat{T}: G \to \hat{U}(H)$. One usually attempts to reduce this nonlinear representation to a linear one by lifting $\hat{T}$ to a representation $T: G \to U(H)$ by unitary operators in H. The problem, therefore, is to find a homomorphism T such that the diagram below commutes, where $\hat{T}$ is the given projective representation:

$$\begin{array}{ccccccccc} 1 & \to & U(1) & \to & U(H) & \xrightarrow{\pi} & \hat{U}(H) & \to & 1. \\ & & & & & \nwarrow^{T} & \uparrow{\scriptstyle \hat{T}} & & \\ & & & & & & G & & \end{array}$$

To be more explicit: suppose that for each element g of G one chooses an element $S(g)$ in $U(H)$ such that $\pi(S(g)) = \hat{T}(g)$. It will not necessarily be the case that $S: G \to U(H)$ is an homomorphism; all one can be sure of is that

$$S(g_1)S(g_2) = \sigma(g_1, g_2)S(g_1g_2) \tag{1}$$

for some complex number $\sigma(g_1, g_2)$ of unit modulus. On the other hand, if κ is any map $G \to U(1)$ and $T: G \to U(H)$ is defined by $T(g) = \kappa(g)S(g)$ then $\pi \circ T = \hat{T}$ also; moreover

$$T(g_1)T(g_2) = \tau(g_1, g_2)T(g_1g_2)$$

where $\tau: G \times G \to U(1)$ is given by

$$\tau(g_1, g_2) = (\kappa(g_1)\kappa(g_2)/\kappa(g_1g_2))\sigma(g_1, g_2). \tag{2}$$

The problem therefore, having chosen S, is to find κ such that $\tau(g_1, g_2) = 1$ for all $g_1, g_2 \in G$. The one available general piece of information about σ comes from the associativity of group multiplication: σ must satisfy, for all $g_1, g_2, g_3 \in G$,

$$\sigma(g_1, g_2)\sigma(g_1g_2, g_3) = \sigma(g_1, g_2g_3)\sigma(g_2, g_3). \tag{3}$$

These are the bare algebraic bones of the problem; in general G will be a topological group and all the maps concerned will be required to be continuous. It is possible to obtain general lifting results from the condition in equation (3) by exploiting assumed topological properties of G; we give an example which is instructive for what follows. Let G be a compact, connected, simply connected Lie group. It is known that in this case there is a globally defined continuous map $S: G \to U(H)$ such that $\pi \circ S = \hat{T}$; *a priori*, however, one knows no more about this map than is contained in (1) and (3), with σ continuous. Suppose that $T: G \to U(H)$ is defined as before. Since G is simply connected, there are well defined continuous functions $\xi, \eta: G \times G \to R$ and $\phi: G \to R$ such that

$$\sigma(g_1, g_2) = \exp[\mathrm{i}\xi(g_1, g_2)], \qquad \tau(g_1, g_2) = \exp[\mathrm{i}\eta(g_1, g_2)], \qquad \kappa(g) = \exp[\mathrm{i}\phi(g)].$$

Then

$$\eta(g_1, g_2) = \phi(g_1) + \phi(g_2) - \phi(g_1g_2) + \xi(g_1, g_2)$$

and the associativity condition (3) reads

$$\xi(g_1, g_2)+\xi(g_1g_2, g_3)=\xi(g_1, g_2g_3)+\xi(g_2, g_3). \tag{4}$$

Since G is a compact Lie group there is an invariant measure μ (Haar measure) defined on it: one may therefore integrate functions defined on G over G. Define

$$\phi(g)=-\int_G \xi(g, h)\,\mathrm{d}\mu(h);$$

then by integrating (4) with respect to g_3 one finds that

$$\xi(g_1, g_2)+\phi(g_1g_2)=\phi(g_1)+\phi(g_2).$$

So with this choice of ϕ one achieves the desired aim, that $\eta=0$, and $\tau=1$. Thus every projective representation of a compact, connected, simply connected Lie group lifts to a unitary representation.

This argument demonstrates the usefulness of integration in solving lifting problems in the compact case.

When such a method is not available the problem may be approached by analysing the properties of G. By using the projective representation $\hat{T}$ one may construct a new group E, called a central extension of G by $U(1)$, as follows:

$$E=\{(g, u)\in G\times U(H)|\hat{T}(g)=\pi(u)\}.$$

The relationship of E to G is much the same as the relationship between $U(H)$ and $\hat{U}(H)$: the map $p_1{:}E\to G$ by $(g, u)\to g$ (projection onto the first factor of $G\times U(H)$ restricted to E) is a homomorphism of E onto G with kernel $U(1)$; if moreover $U(1)$ is injected into G by the map $\theta{:}U(1)\to E$ by $\zeta\to(e, \zeta\cdot 1)$, where e is the identity of G and 1 the identity of $U(H)$, then $\theta(U(1))$ lies in the centre of G; hence the term central extension. The relationship of E and G to the other groups in the problem is best summarised by the following commutative diagram, in which both rows are exact:

$$\begin{array}{ccccccccc}
1 & \to & U(1) & \to & U(H) & \xrightarrow{\pi} & \hat{U}(H) & \to & 1\\
 & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle p_2} & & \uparrow{\scriptstyle \hat{T}} & & \\
1 & \to & U(1) & \xrightarrow[\theta]{} & E & \xrightarrow[p_1]{} & G & \to & e.
\end{array}$$

(The map p_2 is projection onto the first factor of $G\times U(H)$ restricted to E.) If there is a homomorphism $q{:}G\to E$ such that $p_1\circ q$ is the identity the central extension E of G is said to split; if this is the case then $p_2\circ q$ is a lifting of $\hat{T}$. Thus the lifting problem for G may be tackled by investigating the possible central extensions of G by $U(1)$. In particular, if it is known that all such central extensions split it will follow that every projective representation of G lifts.

One further example before we turn to classical mechanics is designed to show that it is sometimes possible to solve the lifting problem by proving a splitting result about the upper sequence. For example, suppose that H is finite dimensional. If $\dim H=n$, we may define a map (a section) $s{:}\hat{U}(H)\to U(H)$ locally by

$$s(\hat{u})=(\det u)^{-1/n}u, \tag{5}$$

where u is any operator in the ray $\hat{u}$. This is well defined, continuous in a neighbourhood of $1\in\hat{U}(H)$ (the nth root being unambiguous, by continuity) and defines a local homomorphism. That is, (5) gives a local splitting of the upper sequence. Now suppose that G is a compact, connected, simply connected group and $\hat{T}: G \to \hat{U}(H)$ is continuous. Thus $\hat{T}$ is a homeomorphism onto its image (because it is a continuous injection from a compact space to a Hausdorff one). In particular, $\hat{T}(G) \subset \hat{U}(H)$ is simply connected, and so the local section defined by (5) may be extended globally over $\hat{T}(G)$ (using the same formula). The conclusion is that finite-dimensional projective representations of such groups G always lift to unitary representations.

The interest of this example in the present context lies not so much in the result, but in the method, in which consideration of the top exact sequence leads to a lifting theorem. The result for compact symplectic manifolds to be proved below works in a similar way.

The phase space of a classical mechanical system is usually taken to be a symplectic manifold (M, Ω). The (infinitesimal) automorphisms of such a system are described by the set $\mathscr{H}(M)$ of globally Hamiltonian vector fields, defined as the image of the set $\mathscr{F}(M)$ of smooth functions on M under the map $\pi: \mathscr{F}(M) \to \mathscr{X}(M)$ (where $\mathscr{X}(M)$ is the set of all smooth vector fields on M) defined by

$$\pi(f) \lrcorner \Omega = -\mathrm{d}f.$$

$\mathscr{F}(M)$ is a Lie algebra with respect to the Poisson bracket, defined by

$$\{f, g\} = \pi(f)g,$$

and $\mathscr{H}(M)$ is a Lie algebra under the usual bracket. Then π is a homomorphism $\pi: \mathscr{F}(M) \to \mathscr{H}(M)$. This gives the exact sequence (of Lie algebra homomorphisms)

$$0 \to R \to \mathscr{F}(M) \xrightarrow{\pi} \mathscr{H}(M) \to 0$$

where R denotes the one-dimensional Lie algebra of constant functions on M. A Lie algebra $\mathscr{G}$ is a symmetry algebra of (M, Ω) if there is an injective homomorphism $\hat{t}: \mathscr{G} \to \mathscr{H}(M)$. One usually attempts to lift $\hat{t}$ to a homomorphism $t: \mathscr{G} \to \mathscr{F}(M)$ since elements of $\mathscr{F}(M)$ are interpreted as observables. Defining $\mathscr{E} = \{(\gamma, f) \in \mathscr{G} \oplus \mathscr{F}(M) | \hat{t}(\gamma) = \pi(f)\}$, one obtains a pair of exact sequences (of Lie algebra homomorphisms)

$$\begin{array}{ccccccccc}
0 & \to & R & \to & \mathscr{F}(M) & \xrightarrow{\pi} & \mathscr{H}(M) & \to & 0 \\
 & & \updownarrow{\scriptstyle id} & & \uparrow{\scriptstyle p_2} & & \uparrow{\scriptstyle \hat{t}} & & \\
0 & \to & R & \longrightarrow & \mathscr{E} & \xrightarrow[p_1]{} & \mathscr{G} & \to & 0.
\end{array}$$

Then the Lie algebra $\mathscr{E}$ is a central extension of $\mathscr{G}$ by R. As in the quantum mechanical case, this lifting problem can be tackled by finding all possible central extensions of $\mathscr{G}$ by R. If all such extensions split, then every $\hat{t}$ can be lifted. However in certain circumstances the problem can be solved by concentrating on the upper exact sequence rather than the lower one.

Assume henceforth that M is compact. Define a section $s: \mathscr{H}(M) \to \mathscr{F}(M)$ (not necessarily a homomorphism) by (for example) the map (linear in X) which assigns, to each $X \in \mathscr{H}(M)$, the function f which vanishes at a given $p_0 \in M$ and satisfies $\pi(f) = X$. Let $v: \mathscr{H}(M) \times \mathscr{H}(M) \to \mathscr{F}(M)$ be defined by

$$v(X, Y) = \{s(X), s(Y)\} - s([X, Y]). \qquad (6)$$

By applying π to (6) we quickly see that each image of v is a constant function. Any other section is given by a linear map t with $t(X)=s(X)+k(X)$ for some linear function $k:\mathscr{H}(M)\to R\subset\mathscr{F}(M)$; the corresponding map $w:\mathscr{H}(M)\times\mathscr{H}(M)\to\mathscr{F}(M)$ is given by

$$w(X, Y)=v(X, Y)-k([X, Y]).$$

We shall show that (6) already implies the existence of a linear function k such that $w=0$; this will imply that the corresponding t defines a homomorphism, so that the upper sequence splits. To establish the required result, we need the following lemma.

Lemma. Let $\Omega^n=\Omega\wedge\Omega\wedge\ldots\wedge\Omega$ (n factors) denote the volume $2n$-form derived from Ω (where $n=\frac{1}{2}\dim M$), and let $f, g\in\mathscr{F}(M)$. Then $\{f, g\}\Omega^n$ is exact.

Proof. Recall that $X\in\mathscr{H}(M)$ implies that $L_X\Omega=0$, where L_X means Lie derivative. Thus

$$L_{\pi(f)}(g\Omega^n)=(L_{\pi(f)}g)\Omega^n=(\pi(f)g)\Omega^n=\{f, g\}\Omega^n. \qquad (7)$$

But the Lie derivative of the $2n$-form $g\Omega^n$ is also given by

$$L_{\pi(f)}(g\Omega^n)=\pi(f)\lrcorner\,\mathrm{d}(g\Omega^n)+\mathrm{d}(\pi(f)\lrcorner\, g\Omega^n). \qquad (8)$$

Since the exterior derivative of any $2n$-form vanishes, the first term on the right-hand side of (8) vanishes; (7) and (8) then give

$$\{f, g\}\Omega^n=\mathrm{d}(\pi(f)\lrcorner\, g\Omega^n).$$

Now we apply the lemma to (6), denoting $\int_M\Omega^n$ by V. Integrating (6) against Ω^n, we obtain

$$v(X, Y)V=\int_M\{s(X), s(Y)\}\Omega^n-\int_M s([X, Y])\Omega^n=-\int_M s([X, Y])\Omega^n,$$

because the first term goes out by exactness. So, choosing the linear function $k:\mathscr{H}(M)\to R$ given by

$$k(X)=-V^{-1}\int_M s(X)\Omega^n$$

we get $w=0$, so t is a homomorphism, so the upper sequence splits. This gives us the following theorem.

Theorem. For any compact symplectic manifold (M, Ω), the exact sequence

$$0\to R\to\mathscr{F}(M)\to\mathscr{H}(M)\to 0$$

splits.

This has the consequence that, for any classical mechanical system described by a compact symplectic manifold, any symmetries represented by a subalgebra of the automorphism algebra $\mathscr{H}(M)$ may be represented by an isomorphic algebra of observables under Poisson bracket.